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LARGE DEFORMATIONS OF RIGID-PLASTIC CIRCULAR PLATES

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Abstract—The large deflections of rigid-plastic circular plates are analyzed by making use of the generalized yield line method which takes into account the changes in geometry of the structures. The circular plates are assumed to deform into a number of right circular cones separated by concentric hinge circles with no radial strains in each cone. Then, the general equation to obtain the load-deflection relations is derived from the principle of virtual velocity.

Simply supported circular plates under circular loading are investigated, the boundary of which is either restrained against inward movement or free to move inward. The results are compared with those obtained by other researchers. The method to account for the elastic deformations is discussed.

NOTATION

- a radius of circular plate
- A, B, C functions of ν
- b radius of circular loading
- $\bar{b}$ b/a
- e $(E/\sigma_0)(t/a)^2$
- E Young's modulus
- F integral defined by eqn (6)
- H rise of shallow conical shell
- k number of hinge circles
- M_γ, M_θ radial and circumferential bending moments
- $\bar{M}_\gamma, \bar{M}_\theta$ M_γ/M_0, M_θ/M_0
- M_0 maximum plastic bending moment $= (1/4)\sigma_0 t^2$
- N_γ, N_θ radial and circumferential membrane forces
- $\bar{N}_\gamma, \bar{N}_\theta$ N_γ/N_0, N_θ/N_0
- N_0 maximum plastic membrane force $= \sigma_0 t$
- p distributed pressure
- $\bar{p}$ uniformly distributed pressure
- P total load
- $\bar{P}$ P/P_0
- P_0 collapse load
- γ, θ, z cylindrical coordinates as shown in Fig. 1
- $\bar{\gamma}$ γ/a
- $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3$ dimensionless radial distances defined by eqns (38), (52) and (53), respectively
- t plate thickness
- u radial displacement
- u_0 coefficient in expression of radial displacement
- w deflection
- w_0 deflection at center of circular plate
- $\bar{w}_0$ w_0/t
- Δ inward displacement at boundary of circular plate
- $\epsilon_\gamma(z), \epsilon_\theta(z)$ radial and circumferential strains at altitude z
- $\epsilon_{\gamma 0}, \epsilon_{\theta 0}$ radial and circumferential membrane strains
- $\kappa_\gamma, \kappa_\theta$ radial and circumferential curvatures
- ν Poisson's ratio
- σ_0 yield stress
- $(\dot{\ })$ derivative with respect to time
- $]$ jump in preceding quantity

1. INTRODUCTION

The behavior of structures may be analyzed reasonably well by rigid-plastic large deflection analysis, in which geometry changes are taken into account while elastic deformations are neglected.

The large deflections of rigid-plastic circular plates under circular loading were investigated by Onat and Haythornthwaite[1]. The deformations at large deflection were assumed to be of the same shape as the incipient velocity at collapse. The boundary of a simply supported plate was free to move inward while that of a rotationally fixed plate was restrained against inward movement. By the upper bound theorem of limit analysis which does not retain large deflection, the collapse load at every stage of deformations was obtained. The analysis was based on the exact yield surface for a uniform plate subjected to the Tresca yield condition.

Hodge[2] used the yield surface for sandwich plate and the upper bound theorem of Onat and Haythornthwaite[1] to analyze a simply supported plate, the boundary of which was free to move inward.

Onat[3] obtained the collapse load of shallow conical shells, from which he derived the load-deflection curves for a simply supported circular plate. The boundary was assumed to be free to move inward. The results were considerably different from those obtained by Onat and Haythornthwaite[1] and Hodge[2].

Rzhanitsyn[4] applied the ganeralized yield line method to rigid-plastic polygonel plates subjected to a concentrated load. It was assumed that the plate deformed into a pyramid with a vertex at the point of the applied load and yield hinges developed along the lateral edges with the rest of the plate remaining rigid. The load-deflection relations were obtained by the principle of stationary potential energy associated with the deformation theory of plasticity and the von Mises yield condition. A simply supported circular plate, the boundary of which was either restrained against inward movement or free to move inward, was analyzed as the limiting case of a polygonal plate.

Sawczuk[5–7] applied the generalized yield line method to analyze a simply supported plate, its edge being either restrained against inward movement or free to move inward. By using the principle of virtual velocity, the general equations to get the load-deflection relations were derived in polar coordinates. The analysis was based on either the flow theory or the deformation theory of plasticity.

Duszek[8] obtained the approximate solution for a simply supported plate under uniform load with the boundary restrained against inward movement. The material was assumed to obey the Tresca yield condition.

Jones[9] obtained the exact solution for a simply supported plate based on the two-moment limited interaction surface[10].

In this paper, large deflection behavior of circular plates under circular loading is investigated using the generalized yield line method which was developed by Rzhanitsyn[4] and Sawczuk[5–7]. At first, the general equation to obtain the load-deflection relations is derived in polar coordinates by making use of the principle of virtual velocity. The procedure to derive the general equation is different from that by Sawczuk[7] although the obtained equation is the same. Then, they are applied to simply supported circular plates. The boundary is assumed to be either restrained against inward movement or free to move inward. Finally the results are compared with relations obtained by other researchers. And the method to account for the elastic deformation of circular plates is discussed.

2. GOVERNING EQUATIONS

We analyze axisymmetric deformations of circular plates under axisymmetric loading. We use polar coordinates γ and θ which lie in the midplane of a circular plate of radius a and thickness t. If the displacement components of a point in the midplane are u and w, *respectively, along* γ and transverse to the plate as shown in Fig. 1, then the strain displacement relations are given by

$$\epsilon_\gamma(z) = \epsilon_{\gamma 0} + z\kappa_\gamma, \quad \epsilon_\theta(z) = \epsilon_{\theta 0} + z\kappa_\theta \tag{1}$$

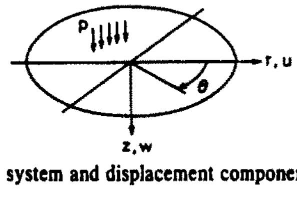

Fig. 1. Coordinate system and displacement components for circular plate.

where

$$\epsilon_{\gamma 0}=\frac{\mathrm{d}u}{\mathrm{d}\gamma}+\frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}\gamma}\right)^2,\quad \epsilon_{\theta 0}=\frac{u}{\gamma}$$

$$\kappa_\gamma=-\frac{\mathrm{d}^2w}{\mathrm{d}\gamma^2},\quad \kappa_\theta=-\frac{\mathrm{d}w}{\gamma\,\mathrm{d}\gamma}. \tag{2a–d}$$

The principle of virtual work

$$\int_0^a (N_\gamma\delta\epsilon_{\gamma 0}+N_\theta\delta\epsilon_{\theta 0}+M_\gamma\delta\kappa_\gamma+M_\theta\delta\kappa_\theta)2\pi\gamma\,\mathrm{d}\gamma=\int_0^a(pw)2\pi\gamma\,\mathrm{d}\gamma \tag{3}$$

gives the following equilibrium equations:

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}(\gamma N_\gamma)-N_\theta=0 \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\left\{\frac{\mathrm{d}}{\mathrm{d}\gamma}(\gamma M_\gamma)-M_\theta+\gamma N_\gamma\frac{\mathrm{d}w}{\mathrm{d}\gamma}\right\}+\gamma p=0. \tag{5}$$

Now, we consider the integral

$$F\equiv\int_0^a\left[\frac{\mathrm{d}}{\mathrm{d}\gamma}\left\{\frac{\mathrm{d}}{\mathrm{d}\gamma}(\gamma M_\gamma)-M_\theta+\gamma N_\gamma\frac{\mathrm{d}w}{\mathrm{d}\gamma}\right\}+\gamma p\right]\dot{w}\,\mathrm{d}\gamma \tag{6}$$

which, by means of integrating by parts, becomes

$$\begin{aligned}F=&\int_0^a p\gamma\dot{w}\,\mathrm{d}\gamma+\int_0^a\left\{\gamma(M_\gamma+wN_\gamma)\frac{\mathrm{d}^2\dot{w}}{\mathrm{d}\gamma^2}+(M_\theta+wN_\theta)\frac{\mathrm{d}\dot{w}}{\mathrm{d}\gamma}\right\}\mathrm{d}\gamma\\&+\left\{\frac{\mathrm{d}}{\mathrm{d}\gamma}(\gamma M_\gamma)-M_\theta+\gamma N_\gamma\frac{\mathrm{d}w}{\mathrm{d}\gamma}\right\}\dot{w}\Big|_{\gamma=a}-\gamma(M_\gamma+wN_\gamma)\frac{\mathrm{d}\dot{w}}{\mathrm{d}\gamma}\Big|_{\gamma=a}.\end{aligned} \tag{7}$$

If the plate is assumed to deform into a number of right circular cones separated by k concentric hinge circles with no radial strains in each cone, eqn (7) becomes

$$\begin{aligned}F=&\int_0^a p\gamma\dot{w}\,\mathrm{d}\gamma+\sum_{i=1}^k\gamma(M_\gamma+wN_\gamma)\frac{\mathrm{d}\dot{w}}{\mathrm{d}\gamma}\Big]_i+\int_0^a(M_\theta+wN_\theta)\frac{\mathrm{d}\dot{w}}{\mathrm{d}\gamma}\mathrm{d}\gamma\\&+\left\{\frac{\mathrm{d}}{\mathrm{d}\gamma}(\gamma M_\gamma)-M_\theta+\gamma N_\gamma\frac{\mathrm{d}w}{\mathrm{d}\gamma}\right\}\dot{w}\Big|_{\gamma=a}-\gamma(M_\gamma+wN_\gamma)\frac{\mathrm{d}\dot{w}}{\mathrm{d}\gamma}\Big|_{\gamma=a}\end{aligned} \tag{8}$$

where] denotes a jump in the preceding quantity.

The Tresca yield condition provides, in each cone

$$\left(\frac{N_\theta}{N_0}\right)^2+\left|\frac{M_\theta}{M_0}\right|=1 \tag{9}$$

and at each hinge circle

$$\left(\frac{N_\gamma}{N_0}\right)^2+\left|\frac{M_\gamma}{M_0}\right|=1 \tag{10}$$

where

$$N_0=\sigma_0 t,\quad M_0=\frac{1}{4}\sigma_0 t^2. \tag{11}$$

Therefore, the flow rule gives

$$\frac{\dot{\epsilon}_{\theta 0}}{|\dot{\kappa}_\theta|} = \frac{t}{2}\left(\frac{N_\theta}{N_0}\right) \tag{12}$$

in each cone and

$$\frac{\dot{\epsilon}_{\gamma 0}}{|\dot{\kappa}_\gamma|} = \frac{t}{2}\left(\frac{N_\gamma}{N_0}\right) \tag{13}$$

at each hinge circle, respectively.

3. SIMPLY SUPPORTED CIRCULAR PLATES UNDER CIRCULAR LOADING

We consider a simply supported circular plate subjected to a uniformly distributed circular loading $\bar{p}$ of radius b as shown in Fig. 2. The boundary conditions are at $\gamma = a$

$$w = 0, \quad M_\gamma = 0, \quad u = -\Delta. \tag{14a–c}$$

From eqns (5) and (14b), we have

$$F + \gamma M_\gamma \frac{\mathrm{d}\dot{w}}{\mathrm{d}\gamma}\bigg|_{\gamma=a} = 0. \tag{15}$$

If we consider a deformation which satisfies eqn (14a), then substitution of eqn (8) into eqn (15) results in

$$\int_0^a p\gamma\dot{w}\,\mathrm{d}\gamma = -\sum_{i=1}^{k} \gamma(M_\gamma + wN_\gamma)\frac{\mathrm{d}\dot{w}}{\mathrm{d}\gamma}\bigg]_i - \int_0^a (M_\theta + wN_\theta)\frac{\mathrm{d}\dot{w}}{\mathrm{d}\gamma}\,\mathrm{d}\gamma. \tag{16}$$

Now, we assume that the plate deforms into a cone with an apex at the center. Then, the deflection is represented by

$$w = w_0\left(1 - \frac{\gamma}{a}\right) \tag{17}$$

which satisfies eqn (14a). Since the radial strain is zero in the cone, eqn (2a) gives

$$\frac{\mathrm{d}u}{\mathrm{d}\gamma} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}\gamma}\right)^2 = 0 \tag{18}$$

which has the solution

$$u = \frac{w_0^2}{2a}\left(1 - \frac{\gamma}{a}\right) - \Delta. \tag{19}$$

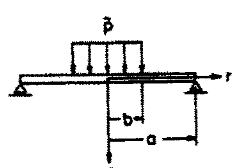

Fig. 2. Simply supported circular plate subjected to uniformly distributed circular loading.

Substitution of eqns (17) and (19) into eqns (2) gives

$$\epsilon_{\gamma 0}=0,\quad \epsilon_{\theta 0}=\frac{w_0^2}{2a\gamma}\left(1-\frac{\gamma}{a}\right)-\frac{\Delta}{\gamma}$$
$$\kappa_\gamma=0,\quad \kappa_\theta=\frac{w_0}{a\gamma}. \tag{20}$$

Therefore, we have

$$\frac{\dot{\epsilon}_{\theta 0}}{\dot{\kappa}_\theta}=w_0\left(1-\frac{\gamma}{a}\right)-a\,\frac{\mathrm{d}\Delta}{\mathrm{d}w_0}. \tag{21}$$

On the other hand, eqn (12) gives

$$\frac{\dot{\epsilon}_{\theta 0}}{\dot{\kappa}_\theta}=\frac{t}{2}\left(\frac{N_\theta}{N_0}\right). \tag{22}$$

It follows from eqns (21) and (22) that

$$\frac{N_\theta}{N_0}=2\left\{\frac{w_0}{t}\left(1-\frac{\gamma}{a}\right)-\frac{a}{t}\frac{\mathrm{d}\Delta}{\mathrm{d}w_0}\right\}. \tag{23}$$

Substitution of eqn (23) into eqn (4), followed by integration, gives

$$\frac{N_\gamma}{N_0}=\frac{2a}{t}\left\{\frac{w_0}{t}\left(1-\frac{\gamma}{2a}\right)-\frac{\mathrm{d}\Delta}{\mathrm{d}w_0}\right\}. \tag{24}$$

In the following, we consider two types of in-plane boundary conditions.

(a) *The case where the boundary is restrained against inward movement.* In this case, we have

$$\Delta=0. \tag{25}$$

Substitution of this equation into eqns (23) and (24) yields

$$\bar{N}_\theta=2\bar{w}_0(1-\bar{\gamma}) \tag{26}$$

$$\bar{N}_\gamma=2\bar{w}_0(1-\bar{\gamma}/2) \tag{27}$$

where

$$\bar{N}_\theta=N_\theta/N_0,\quad \bar{N}_\gamma=N_\gamma/N_0$$
$$\bar{w}_0=w_0/t,\ \bar{\gamma}=\gamma/a. \tag{28}$$

Equations (26) and (27) remain valid provided

$$-1\leq\bar{N}_\theta\leq 1,\quad -1\leq\bar{N}_\gamma\leq 1 \tag{29}$$

which imply

$$\bar{w}_0\leq 1/2. \tag{30}$$

Substitution of eqn (26) into eqn (9) gives

$$\bar{M}_\theta=1-4\bar{w}_0(1-\bar{\gamma})^2 \tag{31}$$

where

$$\bar{M}_\theta = M_\theta / M_0. \tag{32}$$

The stress distributions given by eqns (26), (27) and (31) are depicted in Fig. 3.

Substitution of eqns (17), (26) and (31) into eqn (16) results in

$$\bar{P} = 1 + (4/3)\bar{w}_0^2 \quad \text{when } \bar{w}_0 \leq 1/2 \tag{33}$$

where

$$\bar{P} = P/P_0 \tag{34}$$

in which P and P_0 are the total load and the collapse load, respectively, given by

$$P = \pi b^2 \bar{p}, \ P_0 = 2\pi M_0/\{1 - (2/3)(b/a)\}. \tag{35}$$

When $\bar{w}_0 \geq 1/2$, we have

$$\bar{N}_\theta = 1, \ \ \bar{M}_\theta = 0, \ \ \bar{N}_\gamma = 0 \quad \text{for } 0 \leq \bar{\gamma} \leq \bar{\gamma}_1, \tag{36}$$

and

$$\bar{N}_\theta = 2\bar{w}_0(1-\bar{\gamma}), \ \bar{M}_\theta = 1 - 4\bar{w}_0^2(1-\bar{\gamma})^2$$
$$\bar{N}_\gamma = 2\bar{w}_0\{1 - \bar{\gamma}/2 - \bar{\gamma}_1^2/(2\bar{\gamma})\} \quad \text{for } \bar{\gamma}_1 \leq \bar{\gamma} \leq 1 \tag{37}$$

where

$$\bar{\gamma}_1 = 1 - 1/(2\bar{w}_0). \tag{38}$$

These stress distributions are also shown in Fig. 3. Substitution of eqns (17), (36) and (37) into eqn (16) leads to

$$\bar{P} = 2\bar{w}_0 + 1/(6\bar{w}_0) \quad \text{when } \bar{w}_0 \geq 1/2. \tag{39}$$

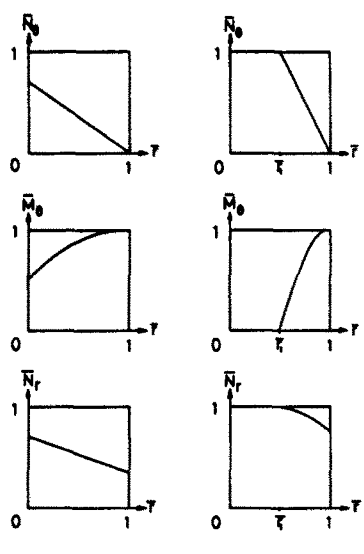

Fig. 3. Stress distribution for simply supported circular plate with boundary restrained against inward movement; (i) when $\bar{w}_0 \leq 1/2$ and (ii) when $\bar{w}_0 \geq 1/2$.

(b) *The case where the boundary is free to move inward.* In this case, the following equation must be satisfied.

$$N_\gamma|_{\gamma=a} = 0. \tag{40}$$

Substitution of eqn (24) into eqn (40) gives

$$\frac{d\Delta}{dw_0} = \frac{1}{2}\frac{w_0}{a} \tag{41}$$

which has the solution

$$\Delta = \frac{1}{4}\frac{w_0^2}{a}. \tag{42}$$

Substitution of eqn (41) into eqns (23) and (24) yields

$$\bar{N}_\theta = 2\bar{w}_0(1/2 - \bar{\gamma}) \tag{43}$$

$$\bar{N}_\gamma = 2\bar{w}_0(1/2 - \bar{\gamma}/2). \tag{44}$$

These equations remain valid provided

$$-1 \leq \bar{N}_\theta \leq 1, \quad -1 \leq \bar{N}_\gamma \leq 1 \tag{45}$$

which imply

$$\bar{w}_0 \leq 1. \tag{46}$$

Substitution of eqn (43) into eqn (9) gives

$$\bar{M}_\theta = 1 - 4\bar{w}_0^2(1/2 - \bar{\gamma})^2. \tag{47}$$

The stress distributions given by eqns (43), (44) and (47) are depicted in Fig. 4. Substitution of eqns (17), (43) and (47) into eqn (16) results in

$$\bar{P} = 1 + (1/3)\bar{w}_0^2 \quad \text{when } \bar{w}_0 \leq 1. \tag{48}$$

When $\bar{w}_0 \geq 1$, the stresses are expressed as

$$\bar{N}_\theta = 1, \quad \bar{M}_\theta = 0, \quad \bar{N}_\gamma = 1 \quad \text{for } 0 \leq \bar{\gamma} \leq \bar{\gamma}_2 \tag{49}$$

$$\bar{N}_\theta = 2\bar{w}_0(1/2 - \bar{\gamma}), \quad \bar{M}_\theta = 1 - 4\bar{w}_0^2(1/2 - \bar{\gamma})^2$$
$$\bar{N}_\gamma = \bar{w}_0(-\bar{\gamma}_2^2/\bar{\gamma} - \bar{\gamma} + 1) \quad \text{for } \bar{\gamma}_2 \leq \bar{\gamma} \leq \bar{\gamma}_3 \tag{50}$$

and

$$\bar{N}_\theta = -1, \quad \bar{M}_\theta = 0, \quad \bar{N}_\gamma = 1/\bar{\gamma} - 1 \quad \text{for } \bar{\gamma}_3 \leq \bar{\gamma} \leq 1 \tag{51}$$

where

$$\bar{\gamma}_2 = 1/2 - 1/(2\bar{w}_0) \tag{52}$$

and

$$\bar{\gamma}_3 = 1/2 + 1/(2\bar{w}_0). \tag{53}$$

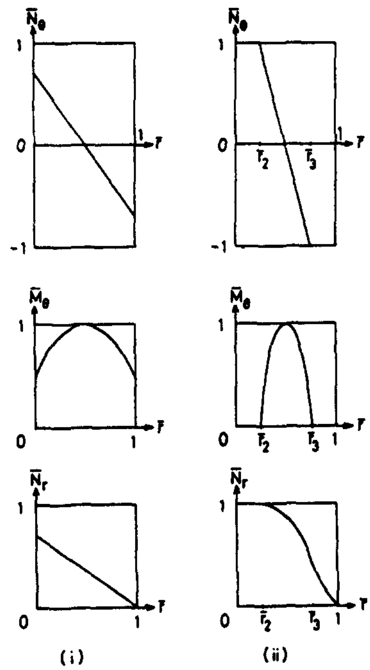

Fig. 4. Stress distribution for simply supported circular plate with boundary free to move inward; (i) when $\bar{w}_0 \leq 1$ and (ii) when $\bar{w}_0 \geq 1$.

These stress distributions are also depicted in Fig. 4. Substitution of eqns (17), (49), (50) and (51) into eqn (16) provides

$$\bar{P} = \bar{w}_0 + 1/(3\bar{w}_0) \quad \text{when } \bar{w}_0 \geq 1. \tag{54}$$

4. RESULTS AND DISCUSSIONS

The load-deflection relations given by eqns (33), (39), (48) and (54) are shown in Fig. 5. It is seen that if the deflections become large, the curves for two types of the in-plane boundary conditions are reduced asymptotically to the straight lines represented, respectively, by

$$\bar{P} = 2\bar{w}_0 \tag{55}$$

and

$$\bar{P} = \bar{w}_0 \tag{56}$$

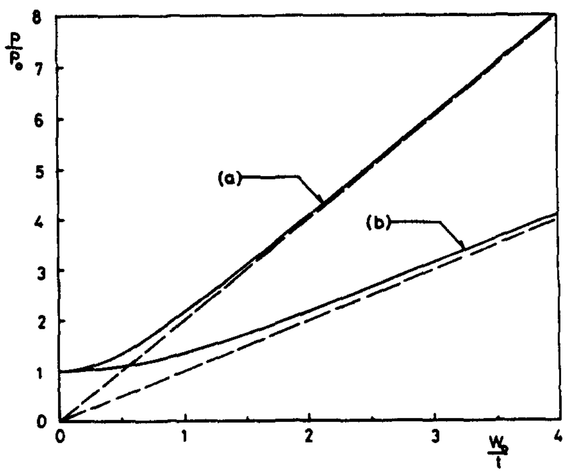

Fig. 5. Load-deflection relations for simply supported circular plate with boundary either (a) restrained against inward movement or (b) free to move inward.

which are indicated by broken lines. It implies that the load of the plate with boundary restrained against inward movement is approximately twice as large as the load of plate with boundary free to move inward for the same amount of deflection. The in-plane boundary conditions have important influence on the behavior after the collapse throughout the entire range of deformations.

The lower parts of both curves, which are represented by eqns (33) and (48), respectively, are the same as those derived by Onat[3] and Sawczuk[7].

However, the curve obtained by Onat and Haythornthwaite[1] for a plate with boundary free to move inward is exactly the same as that given by eqns (33) and (39), which, in the present analysis, are for a plate with boundary restrained against inward movement. Onat and Haythornthwaite[1] assumed that the deformations were given by

$$u = -(1/2)(w_0^2/a)\bar{\gamma}, \quad w = w_0(1-\bar{\gamma}). \tag{57}$$

If we assume that, instead of eqn (57), the deformations are represented by

$$u = -(1/4)(w_0^2/a)\bar{\gamma}, \quad w = w_0(1-\bar{\gamma}) \tag{58}$$

and follow the method developed by Onat and Haythornthwaite[1], then we have the relations

$$\left.\begin{array}{ll} \bar{P} = 1+(1/2)\bar{w}_0+(1/6)\bar{w}_0^2 & \text{when } 0 \leq \bar{w}_0 \leq 1 \\ \bar{P} = 1/2+\bar{w}_0+1/(6\bar{w}_0) & \text{when } 1 \leq \bar{w}_0 \end{array}\right\} \tag{59}$$

which are depicted in Fig. 6. It is seen that the derived curve is considerably different from the original one and close to the result obtained in the present paper. Thus the deformations represented by eqn (58) seem to be more appropriate than those given by eqn (57).

The same comments can be mentioned for Hodge's analysis[2] which is almost the same as Onat and Haythornthwaite analysis[1] except for the yield condition. Hodge applied the assumed deformation given eqn (57) to a plate with boundary free to move inward, obtaining the result

$$\left.\begin{array}{ll} \bar{P} = 1 & \text{when } 0 \leq \bar{w}_0 \leq 1/4 \\ \bar{P} = 2\bar{w}_0+1/(8\bar{w}_0) & \text{when } 1/4 \leq \bar{w}_0 \end{array}\right\} \tag{60}$$

which is almost the same as that of Onat and Haythornthwaite[1]. If we use the assumed

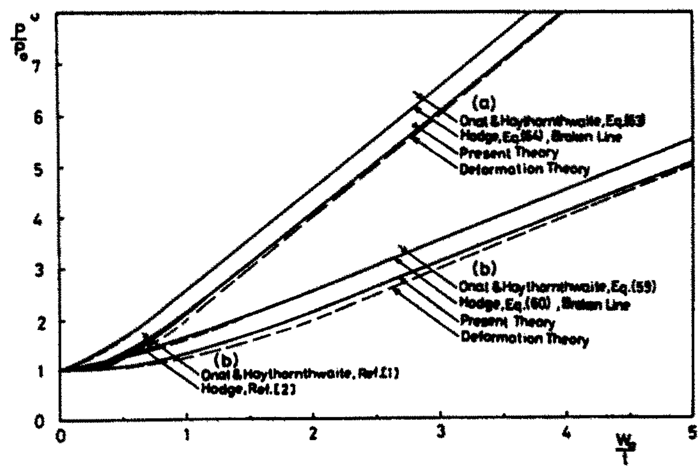

Fig. 6. Comparison of load-deflection relations with those of other researchers for simply supported circular plate with boundary either (a) restrained against inward movement or (b) free to move inward.

deformations given by eqn (58) and follow Hodge's analysis, we have

$$\left.\begin{aligned}\bar{P} &= 1 + (1/2)\bar{w}_0 \quad \text{when } 0 \leq \bar{w}_0 \leq 1/2 \\ \bar{P} &= 1/2 + \bar{w}_0 + 1/(8\bar{w}_0) \quad \text{when } 1/2 \leq \bar{w}_0\end{aligned}\right\} \tag{61}$$

which is close to the relation given by eqn (59).

Thus the results similar to those obtained in this paper can be derived by using the methods of Onat and Haythornthwaite[1] and Hodge[2].

For a plate with boundary restrained against inward movement, if we assume that the deformations are represented as

$$u = 0, \quad w = w_0(1 - \bar{\gamma}) \tag{62}$$

and follow Onat and Haythornthwaite's method[1], then we have the following load deflection relations;

$$\left.\begin{aligned}\bar{P} &= 1 + \bar{w}_0 + (2/3)\bar{w}_0^2 \quad \text{when } 0 \leq \bar{w}_0 \leq 1/2 \\ \bar{P} &= 1/2 + 2\bar{w}_0 + 1/(12\bar{w}_0) \quad \text{when } 1/2 \leq \bar{w}_0\end{aligned}\right\}. \tag{63}$$

And if we follow Hodge's method[2], then we have

$$\left.\begin{aligned}\bar{P} &= 1 + \bar{w}_0 \quad \text{when } 0 \leq \bar{w}_0 \leq 1/4 \\ \bar{P} &= 1/2 + 2\bar{w}_0 + 1/(16\bar{w}_0) \quad \text{when } 1/4 \leq \bar{w}_0\end{aligned}\right\}. \tag{64}$$

Above results are similar to those obtained in the present paper.

Therefore it seems that the results derived for two types of the in-plane boundary conditions in this paper are reasonable.

Rzhanitsyn[4] and Sawczuk[5, 6] analyzed the large deformations of rigid-plastic simply supported plates based on the deformation theory of plasticity and the Tresca yield condition with the results

$$\left.\begin{aligned}\bar{P} &= 1 + \bar{w}_0^2 \quad \text{when } 0 \leq \bar{w}_0 \leq 1 \\ \bar{P} &= 2\bar{w}_0 \quad \text{when } 1 \leq \bar{w}_0\end{aligned}\right\} \tag{65}$$

for a plate with boundary restrained against inward movement, and

$$\left.\begin{aligned}\bar{P} &= 1 + (1/4)\bar{w}_0 \quad \text{when } 0 \leq \bar{w}_0 \leq 2 \\ \bar{P} &= \bar{w}_0 \quad \text{when } 2 \leq \bar{w}_0\end{aligned}\right\} \tag{66}$$

for a plate with boundary free to move inward. The deformations were assumed to be the same as those taken in the present paper for two types of the in-plane boundary conditions, respectively. The load-deflection relations are plotted in Fig. 6. It is seen that the curves are slightly below the corresponding curves based on the flow theory of plasticity.

From the results which have been derived so far, the load-deflection relations for simply supported shallow conical shells with the rise of H as shown in Fig. 7 can be derived. Since a plate has been assumed to deform into a cone, the curve of the relation between the load P/P_0 and the deflection w_0/t for a conical shell is obtained by shifting to the w_0/t direction by an amount equal to H/t as depicted in Fig. 8. When the loads are small, the results are almost the same as those obtained by Onat[3]. It should be noted that the initial slope is not zero.

In the rigid-plastic analyses which have been discussed hitherto, the elastic deformation is disregarded. Now, by assuming that a plate is made of elastic material, we consider the potential energy at large deflections as

$$\begin{aligned}\pi = {} & \frac{Et}{2(1-\nu^2)}\int_0^a (\epsilon_{\gamma 0}^2 + 2\nu\epsilon_{\gamma 0}\epsilon_{\theta 0} + \epsilon_{\theta 0}^2)2\pi\gamma\,\mathrm{d}\gamma \\ & + \frac{Et^3}{24(1-\nu^2)}\int_0^a (\kappa_\gamma^2 + 2\nu\kappa_\gamma\kappa_\theta + \kappa_\theta^2)2\pi\gamma\,\mathrm{d}\gamma - \int_0^b (pw)2\pi\gamma\,\mathrm{d}\gamma\end{aligned} \tag{67}$$

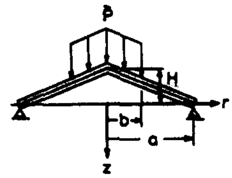

Fig. 7. Simply supported shallow circular conical shell subjected to uniformly distributed circular loading.

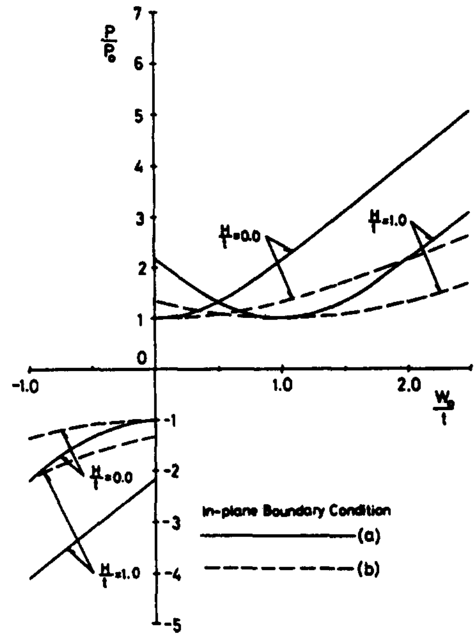

Fig. 8. Load-deflection relations for simply supported shallow circular conical shells with boundary either (a) restrained against inward movement or (b) free to move inward.

where E and ν are Young's modulus and Poisson's ratio, respectively. If we take the assumed deformation field for a plate with boundary restrained against inward movement as

$$\left.\begin{aligned} u &= u_0(\bar{\gamma}^2 - \bar{\gamma}) \\ w &= w_0\left(\frac{1+\nu}{5+\nu}\bar{\gamma}^4 - \frac{2(3+\nu)}{5+\nu}\bar{\gamma}^2 + 1\right) \end{aligned}\right\} \tag{68}$$

and for a plate with boundary free to move inward as

$$\left.\begin{aligned} u &= u_0\left(\bar{\gamma}^2 - \frac{2+\nu}{1+\nu}\bar{\gamma}\right) - \frac{32}{(1+\nu)(5+\nu)^2}\frac{w_0^2}{a}\bar{\gamma}^2 \\ w &= w_0\left(\frac{1+\nu}{5+\nu}\bar{\gamma}^4 - \frac{2(3+\nu)}{5+\nu}\bar{\gamma}^2 + 1\right) \end{aligned}\right\} \tag{69}$$

then, the energy method gives the following results for a plate with boundary restrained against inward movement

$$\bar{P} = \frac{1-(2/3)\bar{b}}{3(5+\nu) - 3(3+\nu)\bar{b}^2 + (1+\nu)\bar{b}^4} \cdot Ae(\bar{w}_0 + B\bar{w}_0^3) \tag{70}$$

and for a plate with boundary free to move inward as

$$\bar{P} = \frac{1-(2/3)\bar{b}}{3(5+\nu) - 3(3+\nu)\bar{b}^2 + (1+\nu)\bar{b}^4} \cdot Ae(\bar{w}_0 + C\bar{w}_0^2). \tag{71}$$

In the above equations, A, B and C are the functions of ν, and

$$e = \frac{E}{\sigma_0}\left(\frac{t}{a}\right)^2, \quad \bar{b} = \frac{b}{a}. \tag{72}$$

The results for $\bar{b} = 1.0$ and $\nu = 0.3$ are shown in Fig. 9 together with the rigid-plastic solution. The real behavior of a circular plate may be well approximated by the elastic curve when the deformation is small, and by the rigid-plastic curve when the deflection is large. However, in the case where e is large, the rigid-plastic analysis gives good approximation throughout the entire range of deflections.

To account for the elastic deformation, Onat and Haythornthwaite[1] proposed the method in which the deflection obtained by the elastic small deflection analysis was added to that obtained by the rigid-plastic large deflection analysis. They showed remarkable agreement between the predicted result and experiment for a simply supported circular plate with boundary free to move inward. The crosses in Fig. 10 show experimental results with elastic linear deflections subtracted, and they agree quite well with the theoretical curve by Onat and Haythornthwaite[1]. However, the agreement does not verify the proposed procedure since their rigid-plastic curve is not considered appropriate as discussed beforehand. As shown in Fig. 10, the proposed curves in this paper show good agreement with the measured test results.

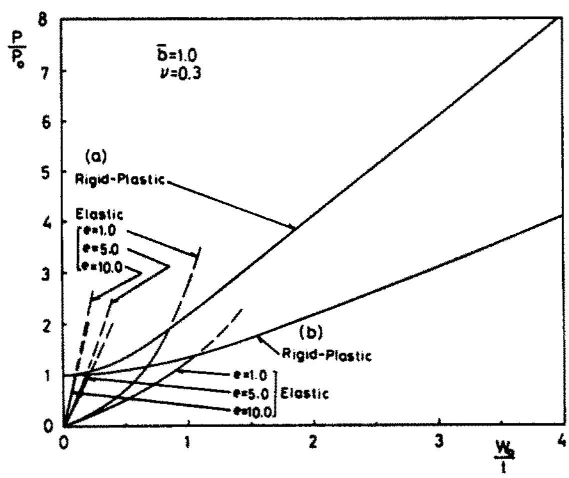

Fig. 9. Elastic solution of load-deflection relations together with rigid-plastic solution for simply supported circular plate with boundary either (a) restrained against inward movement or (b) free to move inward.

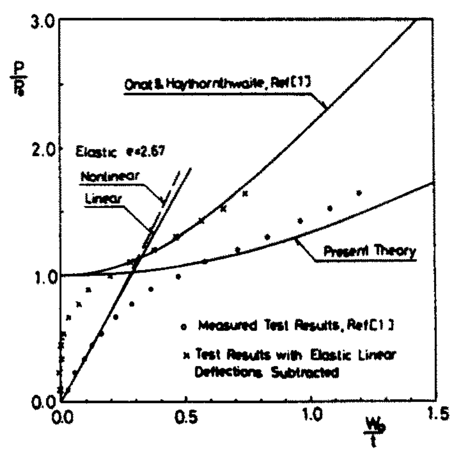

Fig. 10. Experimental and theoretical load-deflection relations for simply supported circular plate with boundary free to move inward.

5. CONCLUSIONS

The generalized yield line method has been formulated in cylindrical coordinates to analyze large deformations of rigid-plastic circular plates under axisymmetric lateral pressure. The procedure to derive the general equation to get the load-deflection relations is different from that by Sawczuk[7] although the obtained equation is the same. Then, it has been applied for simply supported circular plates under circular loading. The boundary has been assumed to be either restrained against inward movement or free to move inward. The obtained load-deflection relations are quite different from those of Onat and Haythornthwaite[1] and Hodge[2]. It has been shown that, if we take appropriate deformations, the methods of Onat and Haythornthwaite[1] and Hodge[2] give almost the same results as those of the present paper. Finally the method to take into account the elastic deformations has been proposed which is different from the procedure suggested by Onat and Haythornthwaite[1]. The proposed load-deflection relation has shown good agreement with test results.

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